



A Short Path to the Shortest Path

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steps of 0.1. This picture inspires one to look for a single equation satisfied by σ_1 and σ_2 . In fact, it is not too hard to show that

$$(1 - \sigma_1)\sigma_2 = \frac{1}{3}. \quad (11)$$

To see this, translate p so that $r_2 = 0$. Then $p'(x) = 3x^2 - 2(r_1 + r_3)x + r_1r_3$. From this, we see that the product of the two roots of p' is $r_1r_3/3$. However, the roots of p' are $(1 - \sigma_1)r_1$ and σ_2r_3 .¹

Figure 2 was produced in a similar manner but using quartic polynomials of the form $p(x) = x(x - r_2)(x - r_3)(x - r_4)$. The critical points were approximated by numerically solving the cubic equation $p'(x) = 0$ using Maple's `fsolve` procedure. This time Y_4 clearly appears to be a smooth surface in X_4 .

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This note contains a demonstration of the isoperimetric inequality. Our proof is somewhat simpler and more straightforward than the usual ones; it is eminently suitable for presentation in an honors calculus course.

1. *The Isoperimetric Inequality* says that a closed plane curve of length 2π encloses an area $\leq \pi$. Equality holds only for a circle.

Let $x(s), y(s)$ be the parametric presentation of the curve, s arclength, $0 \leq s \leq 2\pi$. Suppose that we have so positioned the curve that the points $x(0), y(0)$ and $x(\pi), y(\pi)$ lie on the x -axis, i.e.

$$y(0) = 0 = y(\pi). \quad (1)$$

The area enclosed by the curve is given by the formula

$$A = \int_0^{2\pi} y\dot{x} ds, \quad (2)$$

where the dot denotes differentiation with respect to s . We write this integral as the sum $A_1 + A_2$ of an integral from 0 to π and from π to 2π , and show that each is $\leq \frac{\pi}{2}$.

¹The author thanks the referee for this particularly nice derivation of (11).

According to a basic inequality,

$$ab \leq \frac{a^2 + b^2}{2};$$

equality holds only when $a = b$. Applying this to $y = a$, $\dot{x} = b$, we get

$$A_1 = \int_0^\pi y \dot{x} ds \leq \frac{1}{2} \int_0^\pi (y^2 + \dot{x}^2) ds. \quad (3)$$

Since s is arclength, $\dot{x}^2 + \dot{y}^2 = 1$; so we can rewrite (3) as

$$A_1 \leq \frac{1}{2} \int_0^\pi (y^2 + 1 - \dot{y}^2) ds. \quad (3')$$

Since $y = 0$ at $s = 0$ and π , we can factor y as

$$y(s) = u(s) \sin s, \quad (4)$$

u bounded and differentiable. Differentiate (4):

$$\dot{y} = \dot{u} \sin s + u \cos s.$$

Setting this into (3') gives

$$A_1 \leq \frac{1}{2} \int_0^\pi [u^2(\sin^2 s - \cos^2 s) - 2u\dot{u} \sin s \cos s - \dot{u}^2 \sin^2 s + 1] ds. \quad (5)$$

The product $2u\dot{u}$ is the derivative of u^2 ; integrating by parts changes (5) into

$$A_1 \leq \frac{1}{2} \int_0^\pi (1 - \dot{u}^2 \sin^2 s) ds,$$

clearly $\leq \pi/2$. Equality holds only if $\dot{u} \equiv 0$, which makes $y(s) \equiv \text{constant} \sin s$. Since equality in (3) holds only if $y = \dot{x} = \sqrt{1 - \dot{y}^2}$, $y(s) \equiv \pm \sin s$, $x(s) \equiv \mp \cos s + \text{constant}$. This is a semicircle. Q.e.d.

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A Note on Entire Solutions of the Eiconal Equation

Dmitry Khavinson

The eiconal equation $\sum_{i=1}^n (\partial u / \partial x_i)^2 = 1$, $u: \mathbf{R}^n \rightarrow \mathbf{R}$ is one of the main equations of geometrical optics. Its characteristics represent the light rays, while the level surfaces of solution u can be thought of as wave fronts (cf., e.g., [3]). Here,