DEBT MATURITY IN FINANCIAL NETWORKS*

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Abstract

We develop a model of interbank networks in which banks experience state-contingent liquidity shocks. We show that networks of long-term debt facilitate the efficient transfer of liquidity: They allow shocked banks to raise liquidity using interbank assets as collateral for new debt, diluting interbank liabilities. Networks of long-term debt thus have strikingly different properties from those of short-term, which cannot be diluted; e.g., high indebtness and connectedness can be sources of stability, not fragility. Networks in a specific class, which we call the "exponential networks," implement optimal contingent transfers despite consisting of plain (non-contingent) debt—they are robust but never fragile.

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1 Introduction

Banks are connected in networks of debt. Unlike in the Walrasian model, in which only net positions matter, gross positions are thought to be a source of systemic risk. A number of theory papers support this conclusion, showing, inter alia, that tightly inter-connected network structures are "robust yet fragile," absorbing everyday shocks but amplifying extraordinary ones (see, notably, Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), hereinafter AOT, and Allen and Gale (2000)).

The literature has focused on short-term (one-period) debt, capturing, e.g., repo markets. But, in practice, interbank debts often have longer maturity.[1] Banks maintain these positions even though practitioners and policy makers alike champion the benefits of netting them out, saying, e.g., that "Support for netting is well-nigh universal in the financial industry as well as among policy markers" (Mengle (2010), p. 2).

In this paper, we develop a financial networks model of long-term interbank debt. Do long-term interbank debt networks harbor the same systemic risks as short-term ones? Do the same network structures lead risks to propagate? Do they serve an economic function that could be undermined by netting out debts?

We find that high indebtedness and connectedness can be sources of stability in long-term debt networks, in diametric contrast to short-. Networks in a specific class, which we call the "exponential networks," implement the efficient transfers of liquidity no matter the distribution of shocks. They are robust but never fragile.

One feature of long-term debt underlies our results: It embeds the option to dilute with new debt to a third party. For illustration, consider a bank in the network with debt both from and to other banks that suffers a liquidity shock. If the interbank debts are long term, it can meet the shock by taking on new debt using its debt from other banks as collateral while

[1]Several papers use data on interbank debt in Germany. They find an average maturity longer than a year and a fraction of overnight debt of about 10% (see, e.g., Bluhm, Georg, and Krahnen (2016), Craig and Ma (2021), Craig and Von Peter (2014), Gabrieli and Georg (2014), and Upper and Worms (2004)). Kuo et al. (2014) point to the scarcity of data on the maturity of US interbank debt and develop a method to impute it from payments data, which suggests that about a quarter of it is term debt.

diluting its debt to other banks. Not so with short-term debt, which, being due right away, cannot be diluted—whereas both types of debt networks create collateral on the assets side of bank balance sheets, only short-term debt on the liabilities side encumbers that collateral. The long-term debts impose costs ex post on other banks, whose debts are diluted, but, if the network is appropriately chosen, they could still benefit them ex ante, allowing them to avoid liquidation when they are shocked.

To capture banks' liquidity risk, we employ elements of Holmström and Tirole (1998) in a financial networks model. Banks in the model have a maturity mismatch: They have long-term assets but could suffer a liquidity shock in the short term. We assume that their assets are not perfectly pledgeable. This gives liquidity risk bite: Banks could be unable to raise liquidity by pledging their assets in the market, and could be inefficiently liquidated as a result.

We start with the benchmark of short-term debt networks before turning to our main analysis, of long-term debt. We demonstrate that the benchmark model is isomorphic to AOT's, mutatis mutandis. Thus we can apply their measures of connectedness such as delta connectedness, the bottleneck parameter, and the harmonic distance. From their analysis, we know that (i) netting out short-term debts increases financial stability (Lemma 2); (ii) more connected networks (appropriately defined) are less stable for large shocks (Lemma 3 and Lemma 5); and (iii) there is a "default radius" around a negatively shocked bank, i.e. closely connected banks also default (Lemma 4). Behind all these results is the idea that the necessity of repaying short-term debt impedes banks' financial flexibility, preventing them (or their creditors) from meeting liquidity shocks.

We begin our analysis of the long-term debt network by showing existence and (generic) uniqueness of a payment equilibrium (Proposition 1). We then establish two sets of main results.

The first contrasts the long-term debt network to the short-. We show that, in diametric contrast to what happens with short-term debt, (i) netting out long-term debts undermines financial stability (Proposition 2); (ii) more tightly connected networks are more stable for large shocks (Proposition 3 via delta connectedness and Proposition 5 via the bottleneck parameter); (iii) that there is a "salvation radius" around a not-shocked bank, i.e. closely connected banks are *not* liquidated (Proposition 4 using the harmonic distance).

Although all long-term debt networks enhance financial stability in our model, many are still inefficient, in that more banks than necessary are liquidated. In particular, some banks suck liquidity out of the system only to be liquidated anyway. For example, any symmetric network allocates excess liquidity equally among distressed banks, when a planner would prioritize them, allocating all the liquidity to the largest subset of banks that it can hope to save and writing off the others entirely (Lemma 6).

Our second set of main results pertains to the exponential networks. In these networks, all banks have debts with all others. But these debts are not the same size. Each bank has larger positions with bank B_i than B_{i+1}, a condition we call "assortativity," which creates an endogenous size distribution. We show that if the positions decay exponentially at a high enough rate, the network is the most efficient no matter the distribution of shocks, in the sense that it allows the greatest number of distressed banks to avoid liquidation (Proposition 6). Intuitively, it prioritizes the allocation of liquidity so that the largest bank always gets the liquidity it needs to survive, the second largest does too as long as there is enough left in aggregate after saving the largest bank, and so on. The size distribution is similar to the empirical one, with some banks being "too big to fail" due to only their position in the interbank financial network (as banks are identical in every other way).

Overall, our results provide a new perspective on financial stability. Gross long-term debts can enhance it, suggesting they should not necessarily be netted out—zero-net positions can have positive net present value. Large, highly leveraged banks can facilitate the allocation of liquidity, suggesting they should not necessarily downsize or recapitalize.

The implementation they facilitate reflects practice: All banks maintain gross positions, e.g., cross holdings of loans or bonds, which shocked banks dilute with new senior debt, e.g., repos. (It also points to a possible reason why bankers view the "super senior" bankruptcy treatment of repos as attractive, despite being irrelevant according to the Modigliani–Miller view: It facilitates dilution and therefore contingent liquidity transfers.)

We analyze two extensions that qualify this rosy view of large long-term debts (Section 6). In one, we allow liquidation to be efficient, so there is such a thing as too much financial flexibility; in the other, we assume that any default is costly, not only those that induce inefficient liquidation. In each case, we show how to calibrate debt levels to implement the efficient outcome, albeit only for examples of specific networks. The takeaway is that, per the baseline, debts should be high enough that banks have the flexibility to weather liquidity shocks when liquidation is inefficient, but, now, not so high that it allows them to when it is efficient or that it induces needless defaults.

Our paper makes two main contributions to the literature. First, it shows how maturity matters in financial networks, contributing to the networks literature, which is focused on short-maturity debt.[2] Only three papers study debt maturity in networks models, to our knowledge: (i) Allen, Babus, and Carletti (2012), which, unlike us, focuses on the maturity of debts to investors outside the network, not among banks within it, (ii) Kusnetsov and Veraart (2018), a mathematical finance paper, which studies the problem of defining and constructing the equilibria in the Eisenberg and Noe (2001) environment with multiple maturities, and (iii) He and Li (2022), which studies how maturity transformation via debt chains can mitigate rollover/resale risk. None of these papers studies debt dilution, our main focus. Second, it builds on the idea that default can implement valuable contingencies (notably, Allen and Gale (1998), Dubey, Geanakoplos, and Shubik (1988), and Zame (1993)). We show that the option to dilute provides another layer of contingency on top of the option to default. And we show how networks structures can leverage this option: An appropriately constructed network of plain (non-contingent) debt can in fact implement the (constrained) efficient outcome, allocating all available liquidity to the right set of banks, under fairly general

[2]Surveys include Allen, Babus, and Carletti (2009), Allen and Walther (2021), Glasserman and Young (2016), and Jackson and Pernoud (2021).

conditions.[3] It thus points to an unexplored way time contingency (maturity) substitutes for state contingency, complementing Angeletos's (2002) idea that the set of bonds of all maturities, whose prices depend on the entire term structure of interest rates, can span all assets (see also Gale (1990)).

The rest of the paper proceeds as follows. Section 2 presents the model. Section 3 considers the short-term debt benchmark. Section 4 states the qualitative properties of long-term debt networks. Section 5 analyzes the exponential network. Section 6 analyzes extensions. Section 7 concludes. The Appendix contains all proofs and a table of notations.

2 Model

We consider a model with two dates $t \in \{1, 2\}$ and $N \geq 2$ agents $\mathrm{B}_1, \mathrm{B}_2, ..., \mathrm{B}_N$, which we refer to as "banks." They resemble real-world banks in so far as each has a maturity mismatch. B_i has assets y in place that pay off at Date 2 and, as in Holmström and Tirole (1998), it could suffer a liquidity shock ℓ at Date 1.[4] We assume that only the fraction θ of y is valuable to outsiders and the remaining $(1-\theta)y$ accrues to B_i alone, a formulation that serves as a catch-all for numerous agency problems (see, e.g., DeMarzo and Fishman (2007)). If B_i cannot meet its liquidity shock its assets are thus liquidated/sold for $\theta y < \ell$.

We write $\sigma_i = 1$ if B_i is shocked and $\sigma_i = 0$ otherwise. There is no other risk, so the "state," which is realized at Date 1, is the profile $\{\sigma_i\}_i =: \boldsymbol{\sigma}$. (We impose no restrictions on its distribution.)

In addition to their long-term assets and liquidity needs, banks also have debts maturing at $t = 2$ both to and from other banks. The face value of B_i's debt to B_j is denoted by $F_{i\to j}$, of all its debts to other banks by $F_{i\rightrightarrows} := \sum_{j\neq i} F_{i\to j}$, and of its debt from other banks by $F_{i\leftleftarrows} := \sum_{j\neq i} F_{j\to i}$; $\mathbf{F}_{\rightrightarrows}$ denotes the vector with ith element $F_{i\rightrightarrows}$. The matrix

[3] Probably the strongest condition is that, in our model, each bank's liquidation cost is independent of the state, thus so is the planner's order of priority in saving shocked banks. If it were better to save B_i instead of B_j in some states and B_j instead of B_i in others, the exponential network could be constrained inefficient.

[4] Throughout, we use "liquidity" in the sense of "inside liquidity" in Holmström and Tirole (1998).

$\mathbf{F} := [F_{i\to j}]_{ij}$ defines the interbank network. Following AOT, we assume throughout that it satisfies $F_{i\leftleftarrows} = F_{i\rightrightarrows}$ for all i, so banks have zero net interbank positions. That is a reasonable approximation of reality, as gross interbank positions can be an order of magnitude larger than net.[5]

Banks need not repay the face value of their debt. They can default. If they do, they repay their interbank debts in pro rata shares $F_{i\to j}/F_{i\rightrightarrows} =: \hat{F}_{i\to j}$:

$$R_{i\to j} = \hat{F}_{i\to j} R_{i\rightrightarrows}, \tag{1}$$

where $R_{i\to j}$ is B_i's repayment to B_j and $R_{i\rightrightarrows}$ is its total repayment to all other banks $\sum_{i\neq j} R_{i\to j}$. This assumption follows the literature (Eisenberg and Noe (2001)) and reflects bankruptcy law and practice.[6]

At Date 1, after the shocks $\boldsymbol{\sigma}$ are realized, banks can borrow via debt in a competitive market. We assume that they can issue new debt of high priority, paid ahead of existing interbank debt, reflecting, e.g., how repos are "super-senior" in bankruptcy.[7] They thus can borrow against all their pledgeable assets: The pledgeable part θy of their long-term assets and their debts from other banks, which, being riskless, have value equal to the repayment they receive at Date 2, $R_{i\leftleftarrows} := \sum_{i\neq j} R_{j\to i}$ (we normalize the risk-free rate to zero). B_i is thus liquidated if the total value of its pledgeable assets is less than its liquidity needs, or if

$$\theta y + R_{i\leftleftarrows} < \ell \sigma_i. \tag{2}$$

[5]For example, in 2021, Barclays PLC's net interbank position is about an eighth of its gross, Lloyd's about a fourteenth, and HSBC's about a fifth. Specifically, their loans to and from other banks were, respectively, about 13.9 and 16.4, 7.0 and 7.6, and 83.1 and 101.1 billion GBP; see home.barclays/content/dam/home-barclays/documents/investor-relations/reports-and-events/annual-reports/2021/Barclays-Bank-PLC-2021-AR.pdf, p. 207, www.lloydsbankinggroup.com/assets/pdfs/investors/annual-report/2021/2021-lbg-annual-report.pdf, pp. 207-208, and hsbc.com/investors/results-and-announcements/annual-report, p. 310.

[6]Csóka and Herings (2021) provides an axiomatic foundation for the pro rata assumption.

[7]Thus, there are two priority classes of debt in our model: new repo-type debt paid first and interbank debts paid next pro rata. This is a good approximation of reality, in which the there are two main priority classes: secured debt paid first and unsecured paid next pro rata. (AOT also features two priority classes, but the senior debt is in place at inception.)

In that case its payoff and repayments are zero. The non-pledgeable assets $(1-\theta)y$ are destroyed. That is the only deadweight loss in the model.

Equation (2) captures our key twist relative to the literature: The long-term debt $F_{i\rightrightarrows}$ that B_i has in place *to* other banks does not appear. It does not impede B_i from raising liquidity as short-term debt does (Section 3). That captures how banks suffering liquidity shocks use super-senior (repo) financing to relax their borrowing constraints, something LTCM, Bear Stearns, and Lehman Brothers all did (or tried to).[8]

Banks that are not liquidated at Date 1 continue to produce y at Date 2. B_i's total (real and financial) assets are thus $y-\ell\sigma_i+R_{i\leftleftarrows}$. (The cash raised via new debt at Date 1 and its associated repayment do not appear because, the debt being riskless, the amount borrowed equals the amount repaid, and they cancel out at Date 2; its value accrues only at Date 1, when it could help B_i weather a shock.) At this point, B_i can either default and capture its non-pledgeable asset value $(1-\theta)y$ or repay in full. It thus defaults if

$$\theta y-\ell\sigma_i+R_{i\leftleftarrows}<F_{i\rightrightarrows}. \tag{3}$$

Observe that, unlike liquidation, which destroys non-pledgeable assets, default alone does not cause a deadweight loss, but just a transfer from creditors to debtors. (We include deadweight losses from default in Section 6.2.)

Combining the liquidation condition (2) and the default condition (3) we have the se-

[8]See, e.g., Jorion (2000, pp. 282–284) on LTCM, Rose, Bergstresser, and Lane (2009, pp. 11–13) on Bear, and Valukas (2010 , pp. 3 and 9–10) on Lehman.

quentially rational repayment

$$R_{i\rightrightarrows} = \begin{cases} 0 & \text{if } \theta y - \ell\sigma_i + R_{i\leftleftarrows} \leq 0\,, \\ \theta y - \ell\sigma_i + R_{i\leftleftarrows} & \text{if } \theta y - \ell\sigma_i + R_{i\leftleftarrows} \in (0, F_{i\rightrightarrows}]\,, \\ F_{i\rightrightarrows} & \text{otherwise} \end{cases} \tag{4}$$

$$= \max\left\{0\,,\ \min\left\{\,\theta y - \ell\sigma_i + R_{i\leftleftarrows}\,,\ F_{i\rightrightarrows}\,\right\}\right\}. \tag{5}$$

To define the equilibrium, we also require that markets clear: The repayments B_i receives from other banks coincide with the repayments other banks pay to it:

$$R_{i\leftleftarrows} = \sum_{j\neq i} R_{j\to i}. \tag{6}$$

Definition 1 (Payment Equilibrium)**.** *A* payment equilibrium *is a repayment vector* $\{R_{i\to j}\}_{i\neq j}$ *for each state* $\boldsymbol{\sigma}$ *such that the repayments*

(i) are sequentially rational (equation (5)),

(ii) are paid pro rata (equation (1)), and

(iii) clear the market (equation (6)).

It is convenient to combine the equilibrium conditions to write a vector fixed point equation:

$$\mathbf{R}_{\rightrightarrows} = \left[\min\left\{\mathbf{F}_{\rightrightarrows}\,,\ \hat{\mathbf{F}}^{\top}\mathbf{R}_{\rightrightarrows} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\right\}\right]^{+}, \tag{7}$$

a solution of which is often called the "clearing vector." ($\mathbf{1}$ denotes the vector of N ones: $(1, ..., 1) \in \mathbb{R}^N$.)

The only deadweight loss in the model is due to liquidation. Thus we adopt the following notation of efficiency:

Definition 2 (Efficiency)**.** *One network is more* efficient *than another if fewer banks are liquidated in equilibrium for every state* $\boldsymbol{\sigma}$.

This is a strengthening of AOT's notions of "stability" (fewer liquidations on average for a given number of shocks) and "resilancy" (fewer liquidations in the worst case scenario) in that if one network is more efficient than another it is more stable and more resilient too. The efficiency ranking is not a total order on the set of networks, but we derive strong enough results that it suffices for our purposes.

For several of our results, it is useful to define the following network structures:

Definition 3 (Network typology)**.** *A network is* regular *if* $F_{i\rightrightarrows} = F_{j\rightrightarrows}$ *for* $i \neq j$, symmetric *if* $F_{i\to j} = F_{j\to i}$ *for* $i \neq j$, complete *if* $F_{i\to j}$ *is constant for* $i \neq j$, ring *if it is regular and* $F_{i\to j} = 0$ *unless* $j = i + 1$ *(mod N).*

In words, in a regular network, each bank has the same amount of total debt; in a symmetric network, each pair of banks has zero net positions; in a complete network, every bank has the same debt to every other; in a ring network, each bank has debt to and from one other in a ring.

It is also useful to define several properties of networks, capturing how closely banks are connected to one another.

Definition 4 (Delta connectedness)**.** *A regular network* $\mathbf{F}$ *is* δ*-connected if there is a subset of banks* $\mathscr{B}$ *such that* $\hat{F}_{i\to j} \leq \delta$ *and* $\hat{F}_{j\to i} \leq \delta$ *for all* $i \in \mathscr{B}$ *and* $j \in \mathscr{B}^c$.

It is connected *if it is not* δ*-connected for* $\delta = 0$.

In words, a network has low δ if one of its components has weak ties to the rest of it.

Definition 5 (Harmonic distance)**.** *For a regular network* $\mathbf{F}$*, the* harmonic distance *from* B_i *to* B_j *is the solution to* $d_{i\to j} := 1 + \sum_{k\neq i} d_{i\to k}\hat{F}_{k\to j}$ *for* $i \neq j$ *and* $d_{i\to i} = 0$.

In words, $d_{i\to j}$ is the liability-weighted distance from $i \to j$, which captures how easily liquidity (or the lack thereof) can flow from B_i to B_j.

Definition 6 (Bottleneck parameter)**.** *For a regular network* $\mathbf{F}$*, the* bottleneck parameter *is*

$$\beta = \min_{\mathscr{B}} \sum_{i \in \mathscr{B}} \sum_{j \in \mathscr{B}^c} \frac{\hat{F}_{i \to j}}{|\mathscr{B}||\mathscr{B}^c|}. \tag{8}$$

In words, a network has low β if one of its components has relatively little debt to the rest of it. It is similar to δ above, but directional (in that it is silent about the debt from the rest to the component).

3 Short-term Debt Benchmark

Here we consider the network in which the interbank debts $F_{i \to j}$ are due at Date 1 instead of Date 2. This benchmark both helps us to compare our model to the literature and to contrast our results to those therein.

Now interbank debts, being due immediately, cannot be diluted with new debt at Date 1. Thus B_i is liquidated if its pledgable assets are not sufficient to cover not only its liquidity needs $\ell\sigma_i$ but also its interbank debt $F_{i \rightrightarrows}$, or

$$\theta y + R_{i \leftleftarrows} < \ell\sigma_i + F_{i \rightrightarrows}. \tag{9}$$

This says that B_i is liquidated if the total value of its pledgeable assets is less than its liquidity needs plus its total debt to other banks, i.e. it is liquidated whenever it defaults—with short-term debt, there is no distinction between default and liquidation. (As the liquidation values at Date 1 coincide with the pledgeable value at Date 2—both equal θy—the clearing vector is always the same—defaulting banks repay $R_{i \rightrightarrows} = [\theta y - \ell\sigma_i + R_{i \leftleftarrows}]^+$ for each σ_i. Hence we do not need to adjust the equilibrium definition for the benchmark. Only efficiency changes.)

Without the distinction between liquidation and default, this benchmark boils down to AOT:

Lemma 1 (Isomorphism between benchmark and AOT)**.** *There is an isomorphism between* *equilibrium and welfare in our short-term debt benchmark and AOT's model in the case in which long-term assets are fully destroyed by default and cash holdings are zero (under which they derive most of their results).*

(The isomorphism between our benchmark and AOT, while mechanical, was unexpected to us, as the models seemed different prima facie. Our model seemed to be about liquidity, theirs about solvency—we now see both as about both as, in both, a shock decreases total asset value (solvency) and might not be met only because long-term assets are not fully pledgeable (liquidity).)

We now restate, and sometimes strengthen, several of AOT's results, focusing on those that contrast with our results on the long-term debt network, starting with debt levels.

Lemma 2 (Netting in short-term benchmark)**.** *Suppose* $\mathbf{F}$ *is a regular network.* $\alpha\mathbf{F}$ *is less efficient than* $\mathbf{F}$ *whenever* $\alpha > 1$.

This is a generalization of AOT's Proposition 3 (p. 574) to an arbitrary number of shocked banks (they prove it for just one). It says that less debt is a good thing. Indeed, it would be better to have no debt whatsoever ($\alpha = 0$). Intuitively, when one bank defaults on its debt to another, the other finds it harder to pay its debts to yet another. So distress propagates from shocked banks to otherwise healthy ones, especially when debts are high ($\alpha > 1$).

We now turn to network connectedness.

Lemma 3 (Delta connectedness in short-term benchmark)**.** *Suppose* ℓ *is sufficiently large and exactly one bank is shocked* ($|\boldsymbol{\sigma}| = 1$).

(i) The ring network is the least efficient among all regular networks with $F_{i\rightrightarrows} > (N-1)\theta y$.

(ii) Any regular δ*-connected network with* $N\delta < \theta y/F_{i\rightrightarrows}$ *is strictly more efficient than the ring.*

This strengthens some of the statements in AOT's Proposition 6 (p. 577–578) by adapting them to our notion of efficiency (Definition 2). It says that less connectedness is a good thing.

The ring network, in which every bank has a large exposure to another, is the worst in a class. It is better to weaken the exposures in the sense of lowering delta connectedness. Intuitively, δ captures how much risk can transmit between two components, so, unlike in the ring network, risk cannot spillover from one to another when δ is low.

We now show that there is a "default radius" around a shocked bank.

Lemma 4 (Default radius in short-term benchmark)**.** *Let* $\mathbf{F}$ *be a regular network with* $F_{i\rightrightarrows} \equiv F$ *and suppose that exactly one bank, say* B_j*, is shocked and does not meet its liquidity shocks* ($R_{j\rightrightarrows} = 0$)*. Define* $d^{ST} := \frac{F}{\theta y}$*.*

(i) If $d_{j\to i} < d^{ST}$ *then* B_i *is liquidated.*

(ii) If all banks are liquidated, then $d_{j\to i} < d^{ST}$ *for all* i*.*

This is AOT's Proposition 8 (p. 579). It says that the harmonic distance d is, in a sense, the right measure of an otherwise healthy bank's exposure to a shocked one, in that it captures exactly whether its distress will transmit to it through the network. It defines a radius around a shocked bank within which all banks are liquidated.

AOT link the harmonic distance d to the bottleneck parameter β using Markov chains. They show, roughly, that $d_{i\to j}$ is the mean hitting time of a Markov chain from state i to j and that the bottleneck parameter is closely related to the "conductance" of a graph, which measures how hard it is for a Markov chain on a graph to leave a set of nodes. Hence the next result:

Lemma 5 (Bottleneck connectedness in short-term benchmark)**.** *Suppose the conditions of Lemma 4 are met and that, additionally, the network* $\mathbf{F}$ *is symmetric. Define* $\beta^{ST} := 4\sqrt{\frac{\theta y}{NF}}$ *and* $\beta_{ST} := \min\left\{\frac{\theta y}{2NF}, 1\right\}$*.*

(i) If $\beta > \beta^{ST}$*, then all banks are liquidated.*

(ii) If $\beta < \beta_{ST}$*, then at least one bank is not liquidated.*

This is AOT's Corollary 2 (p. 581). It captures the idea that when all banks are closely connected then risk is so easily transmitted to other banks that a shock at one can lead all to fail, whereas if they are not, it cannot. Specifically, if at least two components are not closely connected, so β is small, risk in one of them cannot spread to the other.

4 Properties of Long-term Debt Networks

We now turn to the properties of networks in our baseline model, with long term debt. For each result in our short-term debt benchmark, we prove a counterpart with opposite sign: Whereas indebtedness and connectness do harm with short-term debt, they do good with long-. The reason is that rather than enabling the liquidity shortage to spread from shocked banks, they allow the liquidity surplus to spread from healthy ones.

We start with existence and uniqueness.

Proposition 1 (Existence and uniqueness)**.** *For any network* $\mathbf{F}$*, a payment equilibrium exists and is generically unique.*

We now turn to debt levels.

Proposition 2 (Netting)**.** *Suppose* $\mathbf{F}$ *is a regular network.* $\alpha\mathbf{F}$ *is more efficient than* $\mathbf{F}$ *whenever* $\alpha > 1$.

This is the counterpart of Lemma 2. It says that *more* debt is a good thing.[9] Intuitively, when one bank needs liquidity, it can pledge its debt from other banks to a third party to raise it. This dilutes its long-term creditors, but allows it to raise liquidity. So liquidity flows from healthy banks to shocked ones, allowing them to weather their shocks.[10]

High debts ($\alpha > 1$) help shocked banks to boost their debt capacity. To see why, suppose two banks, B_i and B_j, have perfectly off-setting debts, owing each other the same amount.

[9] Although more debt cannot hurt if the network structure $\mathbf{F}$ stays the same, it can if it changes—i.e. if some debts increase and some do not—as illustrated by how making a network "more symmetric" can decrease efficiency (in Section 5).

[10] Efficient "dilutable debt" also appears in Diamond (1993), Donaldson, Gromb, and Piacentino (2020, 2021), and Hart and Moore (1995).

If one of them, say B_i, suffers a liquidity shock, it raises liquidity in the market by pledging its own assets and the debt it has from B_j. And the more debt B_i has from B_j, the more it can raise against it.

The off-setting debts with B_j also give B_i more to repay. But that need not impede B_i's raising new debt from the market, which, being senior, transfers the costs of default on to B_j. Meanwhile, B_j has no defense against this dilution. It cannot symmetrically dilute its debt to B_j, because, not being shocked, it repays B_i the same amount regardless of whether it takes on new debt (cf. equation (3)); in particular, there is nothing to gain from dilution absent default—you repay everything anyway.[11] But B_j benefits when it is shocked, symmetrically diluting its debt to B_i. As only whatever bank is shocked exercises its dilution option, liquidity gets transferred from the bank that has liquidity to the bank that needs it: Zero-net long-term debt has positive net present value.

That is not so in the short-term debt benchmark (and most other models), in which zero-net debts have zero NPV at best (Lemma 2). The reason is that short-term debt, being due right away, cannot be diluted: The debt on the asset-side of the balance sheet increases debt capacity but that on the liability side decreases it, at best cancelling each other out.

We turn to network connectedness next.

Proposition 3 (Delta connectedness)**.** *Suppose ℓ is sufficiently large and exactly one bank is* not *shocked ($|\mathbf{1} - \boldsymbol{\sigma}| = 1$).*

(i) The ring network is the most efficient among all regular networks with $F_{i\rightrightarrows} > \theta y$.

(ii) For any δ, there is a δ-connected network that is strictly less efficient than a ring.

This is the counterpart of Lemma 3. It says that *more* connectedness is a good thing. The ring network, in which every bank has a large exposure to another, is the best in a class

[11] This mechanism is self-enforcing, in that, as banks cannot avoid being diluted, they are committed ex ante to transfers they would prefer not to make ex post. Leitner (2005) uncovers another self-enforcing mechanism to transfer liquidity in financial networks: Healthy banks commit to transfer liquidity to distressed by exposing themselves to their default through the network.

(given a single shocked bank; cf. Section 5 on multiple shocks). It is better not to weaken interbank exposures in the sense that making δ small could lead to a strictly worse outcome. Intuitively, δ captures how much liquidity can transmit between two components, so when δ is low, unlike in the ring network, liquidity cannot flow from banks that have it to banks that need it.

We turn to the long-term-debt counterpart to short-term debt's default radius around a shocked bank. It is a "salvation radius" around a healthy bank.

Proposition 4 (Salvation radius)**.** *Let* $\mathbf{F}$ *be a regular network with* $F_{i\rightrightarrows} \equiv F$ *and suppose that exactly one bank, say* B_j*, is* not *shocked and that it does not default. Define* $d^{LT} := \frac{F}{\ell-\theta y}$.

(i) If $d_{j\to i} < d^{LT}$ *then* B_i *is not liquidated.*

(ii) If no bank is liquidated, then $d_{j\to i} < d^{LT}$ *for all* i.

This is the counterpart of Lemma 4. It says that the harmonic distance captures not only how defaults transmit from shocked to healthy banks in short-term debt networks, but also how the option to dilute allows liquidity to flow from healthy to shocked banks in long-term ones. It defines a radius around a not-shocked bank within which *no* bank is liquidated.

AOT's link between the harmonic distance and the bottleneck parameter also applies to our long-term debt network—it relies on only network structures, not equilibrium behavior. Hence we have the next result:

Proposition 5 (Bottleneck connectedness)**.** *Suppose the conditions of Proposition 4 are met and that, additionally, the network* $\mathbf{F}$ *is symmetric. Define* $\beta^{LT} := 4\sqrt{\frac{\ell-\theta y}{NF}}$ *and* $\beta_{LT} := \min\left\{\frac{\ell-\theta y}{2NF}, 1\right\}$.

(i) If $\beta > \beta^{LT}$*, then no bank is liquidated.*

(ii) If $\beta < \beta_{LT}$*, then at least one bank is liquidated.*

This is the counterpart of Lemma 5. It captures the idea that if all banks are closely connected, one bank's excess liquidity can flow through the system to save all banks and, conversely, if they are not, it cannot. Specifically, if at least two components are not closely connected (β is small), the banks in one can raise little liquidity by diluting their debts to banks in the other. What they can raise can be so limited that they end up unable to save themselves from liquidation.

5 Efficiency and the Exponential Network

Here we define constrained efficiency and construct a network to implement it.

Definition 7 (Planner's problem and constrained efficiency)**.** *The* planner's problem *is to find a transfer* $\{t_i\}_i$ *to each* B_i *to minimize the number of liquidated banks* $\big|\{i : \theta y + t_i - \ell\sigma_i < 0\}\big|$ *for each* $\boldsymbol{\sigma}$ *subject to each bank's liquidity constraint* $t_i \geq \min\{\ell\sigma_i - \theta y, 0\}$ *and liquidity being conserved* $\sum t_i \leq 0$*.*

A network is constrained efficient *if the equilibrium is no less efficient than the planner's solution for each* $\boldsymbol{\sigma}$*.*

In words, the planner wants to minimize the number of liquidated banks by transferring liquidity within the system. It must respect the limited pledgeability friction—it cannot raise more than the net liquidity a bank has (that is no more than θy from a not-shocked bank and zero from a shocked one, per the constraint $t_i \geq \min\{\ell\sigma_i - \theta y, 0\}$, given $\ell > \theta y$). The next result characterizes the solution.

Lemma 6 (Constrained efficiency)**.** *A network is constrained efficient if the number of liquidated banks is*

$$L^* := \max\left\{ 0\,,\ \left\lceil \frac{S\ell - N\theta y}{\ell - \theta y} \right\rceil \right\} \tag{10}$$

for each state $\boldsymbol{\sigma}$*, where* S *denotes the number of shocked banks.*

It turns out that the social planner should generally raise as much liquidity as possible from each not-shocked bank, levying the tax $-t_i = \theta y$ if $\sigma_i = 0$, and transfer shocked banks either just enough liquidity to survive or none at all, i.e., either $t_i = \ell - \theta y$ or $t_i = 0$ if $\sigma_i = 1$.

Using S and L to denote the numbers of shocked and liquidated banks, the planner's budget constraint says that the total subsidy—the transfer $\ell - \theta y$ to each of the $S - L$ shocked banks that is not liquidated—must be less than the total tax—the transfer θy from each of the $N - S$ banks that is not shocked:

$$(S - L)(\ell - \theta y) \leq (N - S)\theta y. \tag{11}$$

Solving for the smallest positive integer L that satisfies the above gives the result.

That argument points to two key properties of the planner's solution, both of which help avoid "wasting liquidity":

(i) It extracts the maximum tax from not-shocked banks, since they are not liquidated anyway.

(ii) It gives nothing to liquidated banks, since, analogously, they are liquidated anyway.

We aim to construct a network with both properties (Proposition 6 below). We now build up to it in steps, showing how to achieve one and then the other, starting with the first:

Lemma 7 (High debt mutualizes assets)**.** *Let* $\mathbf{F}$ *be a connected network. If* α *is sufficiently large, then in the equilibrium of* $\alpha\mathbf{F}$ *either (i) all not-shocked banks make the maximum net payment,* $R_{i\rightrightarrows} - R_{i\leftleftarrows} = \theta y$*, or (ii) no bank is liquidated.*

This says that if debts are sufficiently high and liquidity is sufficiently scarce (in the sense that at least one bank is liquidated), then each not-shocked bank provides the maximum amount of liquidity. Intuitively, to increase interbank debts is to make each banks' assets a larger fraction of others' balance sheets—it "mutualizes" the banking system, making each bank more like the whole system.[12] As a result, when liquidity is scarce overall, no surviving

[12]This role of default in facilitating a socially efficient transfer of liquidity contrasts with the literature, in which it typically constitutes a social cost, facilitating rent extraction at best (see Farboodi (2021) and Perotti and Spier (1993)).

bank retains excess liquidity. In contrast, if liquidity is not scarce than no bank is liquidated if debt levels are sufficiently high:

Corollary 1 (First best with high debt for small shocks)**.** *Let* $\mathbf{F}$ *be a connected network and suppose that* $N\theta y > S\ell$ *(i.e.* $L^* = 0$*). If* α *is sufficiently large, then no bank is liquidated in the equilibrium of* $\alpha\mathbf{F}$*.*

The second property—that liquidated banks are transferred nothing—points to how the planner's allocation is necessarily discriminatory: It prioritizes some shocked banks over others. (If in contrast, it allocated the excess liquidity equally among all shocked banks it ends up saving none of them unless it can save them all. As we illustrate in an example below, that symmetry makes the complete network "robust yet fragile," per AOT's result.) That suggests that whenever liquidity is scarce (S is large) a network must be asymmetric to be (constrained) efficient. The following definitions characterize ways in which the network can be asymmetric.

Definition 8 (Assortativity)**.** *A network* $\mathbf{F}$ *is* assortative *if*

$$F_{i\to k} > F_{i\to l} \implies F_{j\to k} > F_{j\to l} \tag{12}$$

for all $i, j, k,$ *and* l *not equal to one another.*

In words, a network is assortative if whenever B_i owes more to one bank than another, so does B_j. Assortativity allows us to rank banks unambiguously by their debt levels. The next definition quantifies/controls that ranking.

Definition 9 (s-dominance)**.** *For a network* $\mathbf{F}$*,* B_i*'s debts are* s-dominated *for* $s \in (0,1)$*, if there is a permutation* π_i *on* $\{1, ..., N\}$ *with* $\pi_i(i) = i$*, such that*

$$\frac{F_{i\to\pi_i(j+k)}}{F_{i\to\pi_i(j)}} \leq s^k \tag{13}$$

for all $i, j,$ *and* $k \geq 0$ *such that* $i \neq j$ *and* $i \neq j + k$*.*

In words, for $s < 1$, B_i's debts to others decay rapidly—its second largest debt is only at most a fraction s of its largest, and so on. (The permutation π_i in the definition just ranks the debts by size.)

Together the definitions above define what we call exponential networks:

Definition 10 (Exponential networks)**.** *A network is an* exponential network *(with base s) if it is connected, assortative, and B_i's debts are s-dominated for all i.*

In words, every bank's debt to B_2 is only at most a fraction s of its debt to B_1 and so on—the permutation π_i in Definition 9 ranks each bank's creditors the same way (assumed w.l.o.g. to be the same as their index ordering per Definition 8). The exponential network generates an approximately exponential distribution of bank asset size, $y + F_{i\rightrightarrows}$.[13,14]

The next two results characterize the payments made to liquidated banks in an exponential network.

Lemma 8 (Controlling relative payments to liquidated banks)**.** *Suppose* $\mathbf{F}$ *is an exponential network with base s and let B_{i^*} be the largest liquidated bank. For each other liquidated bank B_j,*

$$R_{j\leftleftarrows} \leq s^{j-i^*} R_{i^*\leftleftarrows}. \tag{14}$$

This says that when debt levels in the network are exponentially controlled (given s-dominance), so are the repayments to liquidated banks in equilibrium.

The previous result says that s controls the relative payments among liquidated banks. The next says that it controls the total payment to all of them.

Lemma 9 (Controlling total payments to liquidated banks)**.** *Suppose* $\mathbf{F}$ *is an exponential*

[13] The network thus captures the empirical fact that bank size decays rapidly. That is often modeled with a Pareto distribution, which fits the distribution of large banks well. Small banks are smaller than it predicts, however, something in line with the exponential distribution (Janicki and Prescott (2006)).

[14] Another paper in which intermediation networks give rise to an endogenous bank size distribution is Farboodi, Jarosch, and Shimer (2017).

network with base s. If at least one bank is liquidated in equilibrium, then

$$\sum_{i \in \mathscr{L}} R_{i\leftarrow} < \frac{\ell - \theta y}{1 - s}, \tag{15}$$

where $\mathscr{L}$ denotes the set of liquidated banks.

This says that the total transfer to liquidated banks can be made arbitrarily close to a single shocked bank's liquidity shortfall, $\ell - \theta y$, by making s sufficiently small. Intuitively, for s small, the liquidity wasted by transferring it to banks that end up being liquidated anyway would barely have been enough to save even a single one of them (per property (ii) above).

To sum up, if debts are large then no liquidity is wasted on banks that would not have been liquidated anyway (Lemma 7) and if the network is exponential then little is wasted on those that would have been (Lemma 9). The next result builds on these results to show how to achieve constrained efficiency.

Proposition 6 (Efficiency of exponential networks)**.** *Define*

$$s^* := 1 - \frac{\ell - \theta y}{N\theta y - S\ell + (1 + L^*)(\ell - \theta y)}. \tag{16}$$

Let $\mathbf{F}$ be an exponential network with base $s \leq s^*$. For α sufficiently large, $\alpha\mathbf{F}$ is constrained efficient as long as $\frac{S\ell - N\theta y}{\ell - \theta y}$ is not an integer.

This result has a limitation when achieving constrained efficiency requires almost all of liquidity available, i.e. when the the slack in inequality (11) becomes small with $L = L^*$ ($(N - S)\theta y - (S - L^*)(\ell - \theta y) \to 0$). In this case, s^* becomes small so the largest bank in the exponential network is many times larger than the second largest one, and so on. Moreover, in the limit when it equals zero, the network does not achieve constrained efficiency—that is the integer case from the proposition. But the next result suggests this limitation is not too worrisome if we accept a weaker notion of efficiency:

Proposition 7 (Approximate efficiency of exponential networks)**.** *Let* $\mathbf{F}$ *be an exponential network with base* $s \leq 1/2$*. For* α *sufficiently large,* $\alpha\mathbf{F}$ *is "almost constrained efficient" in that at most* $L^* + 1$ *banks are liquidated.*

Finally, we point out that even in the knife-edge case in which the exponential network does not achieve constrained efficiency, no other connected network does either.

Lemma 10 (Inefficiency of other networks)**.** *Suppose* $\frac{S\ell - N\theta y}{\ell - \theta y}$ *is an integer and* $S\ell > N\theta y$ *and that* $\mathbf{F}$ *is fully connected in that* $F_{i\to j} > 0$ *for all* $i \neq j$*. The equilibrium is not constrained efficient.*

This result suggests that the exponential network is the "best" no matter parameters.[15]

Exponential network example. To conclude, we consider an example to illustrate (i) how a complete network leads to inefficient liquidation by allocating liquidity to banks that end up being liquidated in equilibrium and (ii) how an exponential network solves the problem.

The illustration requires that shocks are large enough that at least one bank is liquidated in the constrained-efficient outcome (otherwise a complete network can save all banks). Hence we consider three banks, two of which are shocked: There are $N = 3$ banks with assets $y = 2$, a fraction $\theta = 1/2$ of which is pledgeable. Exactly two banks suffer shocks, $S \equiv \sum \sigma_i = 2$, of size ℓ. We assume that $\ell = 8/5$ so that in the constrained-efficient outcome exactly one bank is liquidated in each state: $\ell < 3\theta y < 2\ell$ (i.e. $8/5 < 3 < 16/5$). Observe that covering either shocked bank's liquidity shortfall requires at least $3/5$ of the other's surplus: $\ell - \theta y = 3\theta y/5$.

We start with the complete network as benchmark and show that both shocked banks are always liquidated. Then we illustrate how the exponential network saves one of them, achieving the constrained-efficient outcome.

Complete network benchmark. Suppose each bank has total debt $F_{i\rightrightarrows} \equiv F$ to others ($F/2$ to each of the other two). For each state $\boldsymbol{\sigma}$, equation (5) gives the system the clearing vector

[15] As the proof implies, a network that is not fully connected could do better for some $\boldsymbol{\sigma}$ but not for all.

$\mathbf{R}_{\rightrightarrows} = [R_{i\rightrightarrows}]_i$ must solve (the full matrix of equilibrium repayments is then given by the pro rata shares $R_{i\to j} = \hat{F}_{i\to j} R_{i\rightrightarrows}$ by equation (1)). When the shocked banks are B_1 and B_2, the system is:

$$\begin{cases} R_{1\rightrightarrows} = \max\Big\{ 0\,,\, \min\big\{ -\frac{3}{5} + \frac{1}{2}R_{2\rightrightarrows} + \frac{1}{2}R_{3\rightrightarrows}\,,\, F \big\} \Big\}, \\ R_{2\rightrightarrows} = \max\Big\{ 0\,,\, \min\big\{ -\frac{3}{5} + \frac{1}{2}R_{1\rightrightarrows} + \frac{1}{2}R_{3\rightrightarrows}\,,\, F \big\} \Big\}, \\ R_{3\rightrightarrows} = \max\Big\{ 0\,,\, \min\big\{ \ 1 + \frac{1}{2}R_{1\rightrightarrows} + \frac{1}{2}R_{2\rightrightarrows}\,,\, F \big\} \Big\}. \end{cases} \tag{17}$$

The first two lines of the system illustrate the problem with the complete network. The repayments from the bank with excess liquidity, B_3, are allocated equally between the two banks with a liquidity shortfall, B_1 and B_2—each gets $\frac{1}{2}R_{3\rightrightarrows}$ (plus a symmetric transfer from the other shocked bank, $\frac{1}{2}R_{2\rightrightarrows}$ or $\frac{1}{2}R_{1\rightrightarrows}$). But each bank needs more than that to survive. Allocating scarce resources equally means that no one has enough.

As the system is symmetric, the problem is analogous when the other pairs of banks are shocked. Solving it gives $R_{i\rightrightarrows} = 0$ if $\sigma_i = 1$ and $R_{i\rightrightarrows} = \min\{F, 1\}$ otherwise, implying that both shocked banks are always liquidated (equation (4)).

Exponential network. Now we turn to an exponential network with base $s = 1/2$:

$$\mathbf{F} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{bmatrix}. \tag{18}$$

Note that the off-diagonal entries in each row and column are decreasing (assortativity per Definition 8) and that each is at most a fraction 1/2 of the previous (s-dominance per Definition 9). We can compute each bank's total debts $F_{i\rightrightarrows}$ (the row sums of $\mathbf{F}$) and the fraction of its payments it makes to each other bank ($\mathbf{F}$ normalized by $F_{i\rightrightarrows}$) :

$$\mathbf{F}_{\rightrightarrows} = \begin{bmatrix} 3 \\ \frac{5}{2} \\ \frac{3}{2} \end{bmatrix} \quad \& \quad \hat{\mathbf{F}}^{\top} = \begin{bmatrix} 0 & \frac{4}{5} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{5} & 0 \end{bmatrix}. \tag{19}$$

It turns out that no matter what pair of banks is shocked, only one is liquidated. To see why, consider the case in which B_1 and B_2 are shocked. In that case, the clearing vector solves (substituting from equation (19) into (5):

$$\begin{cases} R_{1\rightrightarrows} = \max\Big\{\,0\,,\, \min\big\{\, -\frac{3}{5} + \frac{4}{5}R_{2\rightrightarrows} + \frac{2}{3}R_{3\rightrightarrows}\,,\, 3\,\big\}\,\Big\}, \\ R_{2\rightrightarrows} = \max\Big\{\,0\,,\, \min\big\{\, -\frac{3}{5} + \frac{2}{3}R_{1\rightrightarrows} + \frac{1}{3}R_{3\rightrightarrows}\,,\, \frac{5}{2}\,\big\}\,\Big\}, \\ R_{3\rightrightarrows} = \max\Big\{\,0\,,\, \min\big\{\ 1 + \frac{1}{3}R_{1\rightrightarrows} + \frac{1}{5}R_{2\rightrightarrows}\,,\, \frac{3}{2}\,\big\}\,\Big\}. \end{cases} \tag{20}$$

The two lines of the system illustrate how the exponential network allocates liquidity efficiently. The repayments from B_3, the bank with excess liquidity, are allocated primarily to one of the two banks with the liquidity shortfall—B_1 gets $\frac{2}{3}$ of B_3's total repayment, B_2 only $\frac{1}{3}$ of it. And that allows B_1 to survive. B_2 is liquidated. But that is (constrained) efficient as there are not enough resources to save them both anyway.

Solving the system gives the clearing vector $\mathbf{R}_{\rightrightarrows} = \left(\frac{3}{35}, 0, \frac{36}{35}\right)$, which, having only one zero entry, affirms that only B_3 is liquidated (equation (4)).

The other cases are analogous. When B_1 and B_3 are shocked the clearing vector is $\mathbf{R}_{\rightrightarrows} = \left(\frac{3}{7}, \frac{9}{7}, 0\right)$, implying only B_3 is liquidated and when B_2 and B_3 are shocked it is $\mathbf{R}_{\rightrightarrows} = \left(\frac{39}{35}, \frac{1}{7}, 0\right)$, implying only B_3 is.

6 Extensions

Under our baseline assumptions (i) liquidation is always inefficient and (ii) default absent liquidation is costless. Here we relax these assumptions (albeit only for specific networks). We show how to choose debt levels to balance the benefit of high debt in providing flexibility to avoid inefficient liquidation per the baseline with the cost of inducing excessive continuation/default included here.

6.1 Risky Assets and Excessive Flexibility

So far, we assumed that y was sufficiently large that liquidation was always inefficient. Now we assume that y can have any value (but is the same for all banks). Thus it is efficient for all shocked banks to be liquidated if y is low but not if it is high. Here we denote the threshold below which liquidation is efficient by y^* and we show how to choose debt levels in a complete network to implement the efficient liquidation policy.[16]

We consider a complete network with debt levels F, focusing on the case in which $S\ell < N\theta y$ for all y (so, in principle, no bank need be liquidated: $L^* = 0$ in Lemma 6). As the network is symmetric, each shocked/not-shocked bank makes and receives the same payments; we index all shocked banks' payments by s and not-shocked banks' by n. From equation (7), the payment to each type is a pro rata share of the repayment made by all other banks of that type and by all banks of the other type. Hence the equilibrium equations for any y are:

$$\begin{cases} R_{s\rightrightarrows} = \left[\min\left\{ F\,,\, \theta y - \ell + \dfrac{1}{N-1}\Big((S-1)R_{s\rightrightarrows} + (N-S)R_{n\rightrightarrows}\Big)\right\}\right]^+, \\[2ex] R_{n\rightrightarrows} = \left[\min\left\{ F\,,\, \theta y + \dfrac{1}{N-1}\Big(SR_{s\rightrightarrows} + (N-S-1)R_{n\rightrightarrows}\Big)\right\}\right]^+. \end{cases} \tag{21}$$

Solving gives the equilibrium repayments:

$$\begin{cases} R_{s\rightrightarrows} &= \left[F - \frac{N-1}{N-S}(\ell - \theta y)\,\right]^+, \\[2ex] R_{n\rightrightarrows} &= F, \end{cases} \tag{22}$$

with shocked banks being liquidated whenever $F < \frac{N-1}{N-S}(\ell - \theta y)$. Thus the efficient outcome is implemented—banks are liquidated if $y < y^*$ but not if $y \geq y^*$—by setting $F = \frac{N-1}{N-S}(\ell - \theta y^*)$.

[16] If paying ℓ is a deadweight loss, then $y^* = \ell$. If it includes a transfer to unmodeled creditors (like AOT's "outside obligation" v), then $y^* < \ell$. Working with a general y^* allows us to stay agnostic on the interpretation of ℓ.

Intuitively, no matter the value of y, high debts allow banks to maintain flexibility. In the baseline, that is only a good thing, so higher debt can never hurt. Here, it can be bad. But an appropriately chosen debt level implements the efficient outcome no matter the value of y.

6.2 Costly Default

So far, we assumed that, while liquidation entailed a deadweight loss, default was just a transfer. Thus we found that higher debt always (weakly) increased efficiency (Proposition 2). An alternative notion of efficiency could minimize defaults as well as liquidations. Here we show that an exponential network can achieve this goal if the debt levels are not too high, albeit only in an example.

Here we return to the three-bank exponential network in the example in Section 5 and replace the network $\mathbf{F}$ in equation (18) with $\alpha\mathbf{F}$, so increasing α increases indebtedness. Recall that the parameters are such that with two shocked banks the constrained efficient number of liquidations is one. As shocked banks always default (see equation (4)), the constrained efficient number of defaults is two. In the example in Section 5, we achieve the efficient number of liquidations, but not of defaults (all three banks default).

We can achieve both goals by reducing the debt. To see that, set $\alpha = 5/8$ and observe that the clearing vector equilibrium payment is $\mathbf{R}_{\rightrightarrows} = (\frac{1}{40}, 0, \frac{15}{16})$: B_1 is not liquidated and B_3 does not default. (But reducing the debt too much undermines the flexibility to avoid liquidation. If debt is too low, say $\alpha = 1/2$, then the clearing vector is $\mathbf{R}_{\rightrightarrows} = (0, 0, \frac{3}{4})$: Both B_1 and B_2 are liquidated.)

7 Conclusion

We study debt maturity and financial in interbank networks. We show that the right network of long-term debts provides insurance and can even implement the optimal contingent transfers, a finding that stands in contrast to the ideas that debt is a straitjacket and that it entails a fixed payment. It extends the idea that debt embeds contingencies via the option to default by showing that it embeds another option—the option to dilute—and that arranging debts in a network can induce its optimal exercise, even leading to the constrained-efficient outcome in fairly general circumstances.

A Proofs

A.1 Proof of Lemma 1 (Isomorphism between benchmark and AOT)

In AOT, the clearing vector is completely described by (Lemma B2, equation (B3)):

$$\boldsymbol{x} = \left[\min\left\{\, y\mathbf{1}\,,\, \mathbf{Q}\boldsymbol{x} + \boldsymbol{e} + \zeta A\mathbf{1}\,\right\}\right]^{+}. \tag{23}$$

In the case in which long-term assets are fully destroyed in default, the $\zeta = 0$ case, the equation can be re-written as

$$\boldsymbol{x} = \left[\min\left\{\, y\mathbf{1}\,,\, \mathbf{Q}\boldsymbol{x} + \frac{a-v}{a-v+A}(a-v+A)\mathbf{1} - \epsilon\boldsymbol{\sigma}\,\right\}\right]^{+}, \tag{24}$$

having used that, by definition, $\boldsymbol{e} = \boldsymbol{a} - \boldsymbol{v} - \epsilon\boldsymbol{\sigma}$ (with no cash, AOT's $c_i \equiv 0$) and denoting the profile of shock indicator, a notation AOT do not use, by $\boldsymbol{\sigma}$, as in our baseline. This is equivalent to the the equilibrium in our short-term debt benchmark, which per Definition 1 is given by

$$\mathbf{R}_{\rightrightarrows} = \left[\min\left\{\, \mathbf{F}_{\rightrightarrows}\,,\, \hat{\mathbf{F}}^{\top}\mathbf{R}_{\rightrightarrows} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\right\}\right]^{+}, \tag{25}$$

where the color coding represents the mapping between the notation in the two papers, as described in Table 1.

In both models, the banks default whenever they cannot repay the face value of their debts; hence the sets of defaulting banks coincide. Likewise, in both, all defaulting banks are liquidated; hence efficiency coincides too (see Definition 2). $\square$

A.2 Proof of Lemma 2 (Netting in short-term benchmark)

This proof generalizes AOT's proof of their Proposition 3. The idea is to show that in the equilibrium of $\alpha\mathbf{F}$ for $\alpha > 1$, each bank's shortfall $\mathbf{F}_{\rightrightarrows} - \mathbf{R}_{\rightrightarrows}$ is greater than it is in the

Table 1: Notations in AOT and here.

	AOT	This paper
Face value of debt	y_{ji}	$F_{i\to j}$
Payment received	$[\mathbf{Q}\boldsymbol{x}]_i$	$[\hat{\mathbf{F}}^\top \mathbf{R}_{\rightrightarrows}]_i$
Negative shock	$a - v - e_i$	$\ell\sigma_i$
Total assets	$a - v + A$	y
Pledgeable assets	$a - v$	θy
Non-pledgeable assets	A	$(1-\theta)y$

equilibrium of $\mathbf{F}$ and therefore so is the number of defaults.

Lemma A.1. *Define the mapping*

$$\Psi^\alpha : \mathbf{D} \mapsto \Big[\min\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \hat{\mathbf{F}}^\top \mathbf{D} - \theta y \mathbf{1} + \ell\boldsymbol{\sigma}\}\Big]^+. \tag{26}$$

If $\mathbf{R}^\alpha_{\rightrightarrows}$ is a clearing vector of $\alpha\mathbf{F}$, then the "shortfall" $\mathbf{D}^\alpha := \alpha\mathbf{F}_{\rightrightarrows} - \mathbf{R}^\alpha_{\rightrightarrows}$ is a fixed point of Ψ^α.

Proof. We compute, using $\mathbf{Q} \equiv \hat{\mathbf{F}}^\top$:

$$\begin{aligned}
\alpha\mathbf{F}_{\rightrightarrows} - \mathbf{R}^\alpha_{\rightrightarrows} &= \alpha\mathbf{F}_{\rightrightarrows} - \max\big\{\mathbf{0}, \min\{\alpha\mathbf{F}_{\rightrightarrows}, \mathbf{Q}\mathbf{R}^\alpha_{\rightrightarrows} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\}\big\} && \text{(Payment Eqm.)}\\
&= \min\big\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \max\{\mathbf{0}, \alpha\mathbf{F}_{\rightrightarrows} - \mathbf{Q}\mathbf{R}^\alpha_{\rightrightarrows} - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\}\big\} && \text{(Combining)}\\
&= \Big[\min\big\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \alpha\mathbf{F}_{\rightrightarrows} - \mathbf{Q}\mathbf{R}^\alpha_{\rightrightarrows} - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\big\}\Big]^+ && \text{(Interchange min/max)}\\
&= \Big[\min\big\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \mathbf{Q}(\alpha\mathbf{F}_{\rightrightarrows} - \mathbf{R}^\alpha_{\rightrightarrows}) - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\big\}\Big]^+ && \text{(zero-net debt)}
\end{aligned}$$

Substituting from the definitions of $\mathbf{D}^\alpha$ and Ψ^α gives the result. □

Now we show that for $\alpha > 1$, any fixed point of Ψ^α is greater than $\mathbf{D}^1$—i.e. that increasing debt levels increases default:

Lemma A.2. *Let $\mathbf{D}^1$ be a fixed point of Ψ^1 and define*

$$\mathscr{H}^\alpha := \prod_{i=1}^{N} \big[D^1_i, \alpha F_{i\rightrightarrows}\big] \tag{27}$$

For $\alpha > 1$, Ψ^α maps $\mathscr{H}^\alpha$ into itself, i.e. $\Psi^\alpha\big(\mathscr{H}^\alpha\big) \subset \mathscr{H}^\alpha$.

Proof. The upper bound, i.e. that $\Psi^\alpha(\mathbf{D}^\alpha) \leq \alpha \mathbf{F}_{\rightrightarrows}$, follows immediately from the definition of Ψ^α as a minimum.

So we need only to show the lower bound, i.e. that $\Psi^\alpha(\mathbf{D}^\alpha) \geq \mathbf{D}^1$ for all $\mathbf{D}^\alpha \in \mathscr{H}^\alpha$. We have that for $\mathbf{D}^\alpha \in \mathscr{H}^\alpha$, $\mathbf{D}^1 \leq \mathbf{D}^\alpha$ by definition of the domain. Thus we can compute:

$$\begin{aligned} \Psi^\alpha(\mathbf{D}^\alpha) &= \Big[\min\big\{\alpha \mathbf{F}_{\rightrightarrows}\,,\, \mathbf{Q}\mathbf{D}^\alpha - \theta y \mathbf{1} + \ell \boldsymbol{\sigma}\big\}\Big]^+ && (28)\\ &\geq \Big[\min\big\{\alpha \mathbf{F}_{\rightrightarrows}\,,\, \mathbf{Q}\mathbf{D}^1 - \theta y \mathbf{1} + \ell \boldsymbol{\sigma}\big\}\Big]^+ && (29)\\ &\geq \Big[\min\{\mathbf{F}_{\rightrightarrows}, \mathbf{Q}\mathbf{D}^1 - \theta y \mathbf{1} + \ell \boldsymbol{\sigma}\}\Big]^+ && (30)\\ &\equiv \Psi(\mathbf{D}^1) \equiv \mathbf{D}^1, && (31) \end{aligned}$$

since $\mathbf{D}^1$ is a fixed point of Ψ^1 by definition. □

Combining the two lemmata above and applying Brouwer's theorem, we have that for $\alpha > 1$, an equilibrium of $\alpha \mathbf{F}$ is a fixed point of a mapping on $\mathscr{H}^\alpha$. Therefore the generically unique[17] clearing vector, being in $\mathscr{H}^\alpha$, exceeds $\mathbf{D}^1$. □

A.3 Proof of Lemma 3 (Delta connectedness in short-term benchmark)

This proof mirrors AOT's proof of their Proposition 6; we translate it to our notation, adapt it to our notion of efficiency, and add some details.

Throughout we write $F := F_{i\rightrightarrows}$ w.l.o.g., given the network is regular by assumption.

Proof of statement (i). We show, by verification, that in equilibrium all banks are liquidated and, therefore, no network is less efficient than the ring. Assuming all banks are liquidated and letting the single shocked bank be B_1 w.l.o.g., the equilibrium is the solution

[17] Generic global uniqueness follows from the isomorphism to AOT (Lemma 1) and the generic uniqueness of their equilibrium.

to

$$\begin{cases} R_{1\to 2} &= \left[\theta y - \ell + R_{N\to 1}\right]^{+} \\ R_{i\to i+1} &= \theta y + R_{i-1\to i} \qquad\qquad i \in \{2, ..., N\}, \end{cases} \tag{32}$$

with the convention that $N+1 := 1$ for indices. Solving, we have the equilibrium: $R_{1\rightrightarrows} = 0$ and, for $i > 1$, $R_{i\to i+1} = (i-1)\theta y \leq (N-1)\theta y$, which is less than F by hypothesis. I.e., all banks are liquidated.

Proof of statement (ii). Here we show that if $\mathbf{F}$ is δ-connected and δ is small, then not all banks are liquidated and, therefore, the network is more efficient than the ring (in which they are). To do so, we show two lemmata.

Lemma A.3. *Let $\mathscr{B}$ be a subset of banks (not equal to all banks). Suppose that the single shocked bank is not in $\mathscr{B}$. If $\mathbf{F}$ is δ-connected with $\delta < \frac{\theta y}{NF}$, then*

$$\sum_{i\in\mathscr{B}} \sum_{j\in\mathscr{B}^c} R_{i\to j} < \theta y|\mathscr{B}| \tag{33}$$

The result says that the total payment from banks in $\mathscr{B}$ to those in $\mathscr{B}^c$ is small when δ small.

Proof. By the definition of delta-connectedness, $F_{i\to j} \leq \delta F, \forall (i,j) \in \mathscr{B} \times \mathscr{B}^c$. Thus $R_{i\to j} \leq F_{i\to j} \leq \delta F, \forall i \in \mathscr{B}, j \in \mathscr{B}^c$ Now summing i over $\mathscr{B}$, j over $\mathscr{B}^c$ and using $\delta < \frac{\theta y}{NF} < \frac{\theta y}{|\mathscr{B}^c|F}$, gives

$$\sum_{i\in\mathscr{B}} \sum_{j\in\mathscr{B}^c} R_{i\to j} \leq \delta F|\mathscr{B}||\mathscr{B}^c| < \theta y|\mathscr{B}|. \tag{34}$$

□

Lemma A.4. *Maintain the assumptions that the single shocked bank is not in $\mathscr{B}$ and that* $\mathbf{F}$ *is δ-connected with $\delta < \frac{\theta y}{NF}$. If inequality* (33) *holds, then not all banks in banks in $\mathscr{B}$ are liquidated.*

Proof. Suppose, in anticipation of a contradiction, that all banks in $\mathscr{B}$ are liquidated. Thus for each B_j in $\mathscr{B}$, $R_{j\rightrightarrows} = R_{j\leftleftarrows} + \theta y$. Summing over banks in $\mathscr{B}$, we have that $\sum_{j\in\mathscr{B}} R_{j\rightrightarrows} =$

$\sum_{j\in\mathscr{B}} R_{j\leftleftarrows} + \theta y|\mathscr{B}|$. Now just expand $R_{j\rightrightarrows}$ and $R_{j\leftleftarrows}$ into their component payments,

$$\sum_{j\in\mathscr{B}}\left(\sum_{i\in\mathscr{B}} R_{j\to i} + \sum_{i\in\mathscr{B}^c} R_{j\to i}\right) = \sum_{j\in\mathscr{B}}\left(\sum_{i\in\mathscr{B}} R_{i\to j} + \sum_{i\in\mathscr{B}^c} R_{i\to j}\right) + \theta y|\mathscr{B}|, \tag{35}$$

and cancel $\sum_{j\in\mathscr{B}}\sum_{i\in\mathscr{B}} R_{j\to i}$ to get that

$$\sum_{j\in\mathscr{B}}\sum_{i\in\mathscr{B}^c} R_{j\to i} = \sum_{j\in\mathscr{B}}\sum_{i\in\mathscr{B}^c} R_{i\to j} + \theta y|\mathscr{B}| \geq \theta y|\mathscr{B}| \tag{36}$$

This contradicts equation (33), which says that the total payment from $\mathscr{B}$ to $\mathscr{B}^c$ is small.

$\square$

A.4 Proof of Lemma 4 (Default radius in short-term benchmark)

The result is that the same as AOT's. Hence we omit the proof.

A.5 Proof of Lemma 5 (Bottleneck connectedness in short-term benchmark)

Although the result is close to AOT's Proposition 8, the proof is new, as AOT do not include one.

We begin with a lemma that connects the bottleneck parameter β to the harmonic distance d:

Lemma A.5. *Let* $\mathbf{F}$ *be a symmetric financial network of* N *banks. The bottleneck parameter* β *satisfies*

$$\frac{1}{2N\beta} \leq \max_{i,k\,:\,i\neq k} d_{k\to i} \leq \frac{16}{N\beta^2} \tag{37}$$

Proof. This is AOT's Lemma 1 (p. 580). Hence we omit the proof.

$\square$

With this we proceed to the two statements of the lemma.

Proof of statement (i). For $\beta > \beta^{ST} \equiv 4\sqrt{\frac{\theta y}{NF}}$, we have, from Lemma A.5, that

$$d_{j\to i} \leq \frac{16}{N\beta^2} < \frac{16}{N(\beta^{ST})^2} = \frac{F}{\theta y} \equiv d^{ST} \quad \text{for all} \quad i \neq j, \tag{38}$$

per the definition of d^{ST} in Lemma 4. That lemma implies that when B_j is shocked, then each B_i defaults.

Proof of statement (ii). For $\beta < \beta_{ST} \equiv \min\left\{\frac{\theta y}{2NF}\,,\,1\right\}$, we have, from Lemma A.5, that, for some i,

$$d_{j\to i} \geq \frac{1}{2N\beta} > \frac{1}{2N\beta_{ST}} = \max\left\{\frac{F}{\theta y}\,,\,\frac{1}{2N}\right\} \equiv \max\left\{d^{ST}\,,\,\frac{1}{2N}\right\} \geq d^{ST}, \tag{39}$$

per the definition of d^{ST} in Lemma 4. That lemma implies that when B_j is shocked, then B_i does not default. □

A.6 Proof of Proposition 1 (Existence and uniqueness)

Per Section 3, the clearing vector satisfies the same equations in the long- and short-term debt network (only the sets of liquidated banks are different). Thus, given Lemma 1, existence of and generic uniqueness of the clearing vector follow from the analogous results in AOT (their Proposition 1, p. 572). □

A.7 Proof of Proposition 2 (Netting)

The proof is similar to that of Lemma 2, but simpler because we work with repayments $\mathbf{R}_{\rightrightarrows}$ directly instead of "shortfalls" $\mathbf{F}_{\rightrightarrows} - \mathbf{R}_{\rightrightarrows}$.

Define the mapping

$$\Phi^{\alpha}(\mathbf{R}) := \Big[\min\left\{\alpha\mathbf{F}_{\rightrightarrows}\,,\,\mathbf{QR} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\right\}\Big]^{+}. \tag{40}$$

Keeping in mind that $\mathbf{Q} \equiv \hat{\mathbf{F}}^{\top}$, the fixed point of Φ^1 is a clearing vector of $\mathbf{F}$.

Lemma A.6. *Let $\mathbf{R}^1_{\rightrightarrows}$ be a fixed point of Φ^1 and define*

$$\mathscr{I}^\alpha := \prod_{i=1}^{N} \left[R^1_{i\rightrightarrows}, \alpha F_{i\rightrightarrows}\right]. \tag{41}$$

For $\alpha > 1$, Φ^α maps $\mathscr{I}^\alpha$ into itself, i.e. that $\Phi^\alpha(\mathscr{I}^\alpha) \subset \mathscr{I}^\alpha$.

Proof. The upper bound, i.e. that $\Phi^\alpha(\mathbf{R}^\alpha_{\rightrightarrows}) \leq \alpha \mathbf{F}_{\rightrightarrows}$, follows immediately from the definition of Φ^α as a minimum.

So we need only to show the lower bound, i.e. that $\Phi^\alpha(\mathbf{R}^\alpha_{\rightrightarrows}) \geq \mathbf{R}^1_{\rightrightarrows}$. For $\mathbf{R}^\alpha_{\rightrightarrows} \in \mathscr{I}^\alpha$, we can compute:

$$\begin{aligned}
\Phi^\alpha(\mathbf{R}^\alpha_{\rightrightarrows}) &= \left[\min\left\{\alpha \mathbf{F}_{\rightrightarrows}\, , \, \mathbf{Q}\mathbf{R}^\alpha_{\rightrightarrows} + \theta y \mathbf{1} - \ell \boldsymbol{\sigma}\right\}\right]^+ &(42)\\
&\geq \left[\min\left\{\alpha \mathbf{F}_{\rightrightarrows}\, , \, \mathbf{Q}\mathbf{R}^1_{\rightrightarrows} + \theta y \mathbf{1} - \ell \boldsymbol{\sigma}\right\}\right]^+ &(43)\\
&\geq \left[\min\left\{\mathbf{F}_{\rightrightarrows}\, , \, \mathbf{Q}\mathbf{R}^1_{\rightrightarrows} + \theta y \mathbf{1} - \ell \boldsymbol{\sigma}\right\}\right]^+ &(44)\\
&\equiv \Phi^1(\mathbf{R}^1_{\rightrightarrows}) \equiv \mathbf{R}^1_{\rightrightarrows}, &(45)
\end{aligned}$$

since $\mathbf{R}^1_{\rightrightarrows}$ is a fixed point of Φ by definition.

□

Given the lemma, we can apply Brouwer's theorem, to conclude that for $\alpha > 1$, an equilibrium of $\alpha \mathbf{F}$ is a fixed point of a mapping on $\mathscr{I}^\alpha$. Therefore the clearing vector, being in $\mathscr{I}^\alpha$, exceeds $\mathbf{R}^1_{\rightrightarrows}$: All repayments are higher when α is higher, so there are fewer liquidations. □

A.8 Proof of Proposition 3 (Delta connectedness)

Note that this proof makes use of the minimum number of liquidated banks, L^*, derived in Lemma 6, even though that result comes later in the text.

Throughout we assume w.l.o.g. that the B_1 is the not-shocked bank.

Proof of statement (i). Equation (4) and the fact that a ring network is regular (so no shocked bank can repay in full) imply that any shocked bank ($i \geq 2$) repays

$$R_{i\rightrightarrows} \equiv R_{i\to i+1} = \left[\theta y - \ell + R_{i\leftleftarrows}\right]^+ \tag{46}$$

$$= \left[\theta y - \ell + R_{i-1\to i}\right]^+, \tag{47}$$

having used the definition of the ring network (Definition 3).

The expression for $R_{i\rightrightarrows}$ implies that if B_{i-1} is liquidated, then B_i is too. Thus the number of banks that are not liquidated is the maximum index i for which $R_{i-1\to i} \geq \ell - \theta y$.

We can now expand the condition recursively for any B_i that is not liquidated:

$$\ell - \theta y \leq R_{i-1\to i} = \theta y - \ell + R_{i-2\to i-1} \tag{48}$$

$$= k(\theta y - \ell) + R_{i-(k+1)\to i-k} \quad (\text{for } k \in \{1, ..., i-2\}) \tag{49}$$

$$= (i-2)(\theta y - \ell) + R_{1\to 2}. \tag{50}$$

So the number of banks that are not liquidated is

$$\max\left\{ i \leq N\,;\, i - 1 \leq \frac{R_{1\to 2}}{\ell - \theta y} \right\} = \min\left\{ N\,,\, \left\lfloor 1 + \frac{R_{1\to 2}}{\ell - \theta y} \right\rfloor \right\}. \tag{51}$$

The number of liquidated banks is N minus the above:

$$L = \max\left\{ 0\,,\, \left\lceil N - 1 - \frac{R_{1\to 2}}{\ell - \theta y} \right\rceil \right\}. \tag{52}$$

For $F > \theta y$, per the condition of the proposition, $R_{1\to 2} = \theta y$ and the minimum number of liquidations is attained $L = L^*$ (equation (10) with $S = N - 1$).

Statement (ii). A trivial example suffices: The "linkless" network—$F_{i\to j} = 0$ for all i and j—is δ-connected for any δ; in it, all shocked banks are liquidated. That is less efficient than the ring network with $F \geq \theta y$, per statement (i). $\square$

A.9 Proof of Proposition 4 (Salvation radius)

We prove the two statements in turn. The arguments build on AOT's proof of their Proposition 8 (pp. 602–603). Ours are a bit more complicated because we cannot work with the clearing vector $\mathbf{R}_{\rightrightarrows}$, but have to work with the shortfall $\mathbf{D} \equiv \mathbf{F}_{\rightrightarrows} - \mathbf{R}_{\rightrightarrows}$ instead.

Statement (i). The proof comprises (somewhat involved) calculations using the shortfall $\mathbf{D}$, which ultimately allow us to bound the harmonic distance of the not-shocked banks to any liquidated bank.

Throughout we denote the set of defaulting banks by $\mathscr{D}$, that of liquidated banks by $\mathscr{L}$, and, hence, that of those that default but are not liquidated by $\mathscr{D} \setminus \mathscr{L}$. $\mathbf{1}$ denotes the vector of all ones and $\mathbf{I}$ the identity matrix, each of appropriate dimension determined by the context. Given the network is regular, we can write $F_{i\rightrightarrows} \equiv F$.

We start with two lemmata. Each takes as its starting point the equilibrium equation for the shortfall

$$\mathbf{D} = \Big[\min\{\mathbf{F}_{\rightrightarrows}\,,\, \mathbf{Q}\mathbf{D} - \theta y \mathbf{1} + \ell\boldsymbol{\sigma}\}\Big]^{+}, \tag{53}$$

which follows from setting $\alpha = 1$ in Lemma A.1. The first lemma develops the equation for banks that default but are not liquidated; the second for banks that are liquidated.

Lemma A.7. $\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} = \big(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\big)^{-1}\big(\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}\mathbf{1}F - (\theta y - \ell)\mathbf{1}\big)$.

Proof. For banks that default, the shortfall is not zero, and for those that are not liquidated, it is less than F. Hence for B_i in $\mathscr{D}\setminus\mathscr{L}$, the second term under the "min" in equation (53) is the relevant one; writing that elementwise gives the following:

$$\begin{aligned} D_i &= \sum_{k=1}^{N} Q_{ik} D_k - \theta y + \ell & (54)\\ &= \sum_{k\in\mathscr{L}} Q_{ik} D_k + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik} D_k + \sum_{k\notin\mathscr{D}} Q_{ik} D_k - \theta y + \ell & (55)\\ &= \sum_{k\in\mathscr{L}} Q_{ik} D_k + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik} D_k - \theta y + \ell, & (56) \end{aligned}$$

having used that shortfall is zero for banks that do not default ($D_k = 0$ for $k \notin \mathscr{D}$). Re-writing the above in block matrix form, using that fact that liquidated banks repay nothing ($D_k = F$ for $k \in \mathscr{L}$), and rearranging gives:

$$\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} = \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}\mathbf{D}_{\mathscr{L}} - (\theta y - \ell)\mathbf{1} \tag{57}$$

$$= \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}F\mathbf{1} - (\theta y - \ell)\mathbf{1} \tag{58}$$

$$= (\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}(\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}F\mathbf{1} - (\theta y - \ell)\mathbf{1}), \tag{59}$$

where the last expression comes from solving for $\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}}$ and rearranging. □

Lemma A.8. $\big(\mathbf{I} + \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}\big)\big(\ell - \theta y\big)\mathbf{1} > \tilde{\mathbf{Q}}F\mathbf{1}$, *where*

$$\tilde{\mathbf{Q}} := \big(\mathbf{I} - \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}} - \mathbf{Q}_{\mathscr{L},\mathscr{L}}\big). \tag{60}$$

Proof. Banks that are liquidated repay zero (equation (4)), so

$$F < \sum_{k=1}^{N} Q_{ik}D_k - \theta y + \ell \tag{61}$$

$$= \sum_{k\in\mathscr{L}} Q_{ik}D_k + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik}D_k + \sum_{k\notin\mathscr{D}} Q_{ik}D_k - \theta y + \ell \tag{62}$$

$$= \sum_{k\in\mathscr{L}} Q_{ik}D_k + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik}D_k - \theta y + \ell \tag{63}$$

$$= \sum_{k\in\mathscr{L}} Q_{ik}F + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik}D_k - \theta y + \ell, \tag{64}$$

having used that shortfall is zero for banks that do not default ($D_k = 0$ for $k \notin \mathscr{D}$) and F for those that are liquidated ($D_k = F$ for $k \in \mathscr{L}$). The above can be re-written in block-matrix notation, $F\mathbf{1} < \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} + \mathbf{Q}_{\mathscr{L},\mathscr{L}}F\mathbf{1} - (\theta y - \ell)\mathbf{1}$, so the expression for $\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}}$ from Lemma A.7 can be substituted in to get:

$$\mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}(\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}F\mathbf{1} - (\theta y - \ell)\mathbf{1}) + \mathbf{Q}_{\mathscr{L},\mathscr{L}}F\mathbf{1} - (\theta y - \ell)\mathbf{1} > F\mathbf{1} \tag{65}$$

Rearranging the above gives the expression in the lemma. □

Now we now compute a bound on the harmonic distance d. First we use the definition of d (Definition 5) to write in block matrix form:

$$\begin{cases} \mathbf{d}_{j\to\mathscr{L}} = \mathbf{1} + \mathbf{Q}_{\mathscr{L},\mathscr{L}}\mathbf{d}_{j\to\mathscr{L}} + \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}}, \\ \mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}} = \mathbf{1} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}\mathbf{d}_{j\to\mathscr{L}} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}}, \end{cases} \tag{66}$$

where $\mathbf{d}_{j\to\mathscr{L}}$ and $\mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}}$ are vectors that capture the harmonic distances from B_j to each of (i) the liquidated and (ii) the defaulting but not liquidated banks, respectively (cf. equations (B19) and (B20) in AOT). (NB: As, by hypothesis, B_j is the only not-shocked there are no additional terms to not defaulting banks.)

Solving for the system in equation (66)—solving for $\mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}}$ in the second equation and substituting it into the first—gives

$$\mathbf{d}_{j\to\mathscr{L}} = \mathbf{1} + \mathbf{Q}_{\mathscr{L},\mathscr{L}}\mathbf{d}_{j\to\mathscr{L}} + \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}(\mathbf{1} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}\mathbf{d}_{j\to\mathscr{L}}) \tag{67}$$

or, given the definition of $\tilde{\mathbf{Q}}$ in equation (60), $\tilde{\mathbf{Q}}\mathbf{d}_{j\to\mathscr{L}} = (\mathbf{I} + \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1})\mathbf{1}$. From here, we can use Lemma A.8 to write

$$\tilde{\mathbf{Q}}\mathbf{d}_{j\to\mathscr{L}} > \tilde{\mathbf{Q}}\frac{F}{\ell - \theta y}\mathbf{1} \tag{68}$$

As $\tilde{\mathbf{Q}}$ is invertibe and elementwise non-negative,[18] this says that if B_i is liquidated, then

$$d_{j\to i} \geq \frac{F}{\ell - \theta y} \equiv d^{LT}, \tag{69}$$

[18]The result follows Theorem 2 of Plemmons (1977) and exercise 5.8 of Berman and Plemmons (1979, p. 159) given that $\tilde{\mathbf{Q}}$ is the Schur complement of the non-singular M-matrix

$$\begin{bmatrix} \mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}} & -\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}} \\ -\mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}} & \mathbf{I} - \mathbf{Q}_{\mathscr{L},\mathscr{L}} \end{bmatrix}.$$

per the definition of d^{LT} in the proposition. That is the desired result.

Statement (ii). As, by hypothesis, no bank is liquidated, we have that for any B_i $D_i < F_{i\rightrightarrows}$, from the definition of the shortfall $\mathbf{D}$. Thus, from the equilibrium equation for the shortfall (Lemma A.1 with $\alpha = 1$) and the observation that no shocked bank repays in full, $D_i > 0$ (equation (4) given the assumption that banks have zero net positions), we have

$$D_i = \sum_{k \neq i} Q_{ik} D_k + \ell - \theta y, \tag{70}$$

for any B_i for $i \neq j$, where, remember, B_j is the not-shocked bank. Dividing both sides of equation (70) by $\ell - \theta y$ says that $D_i/(\ell - \theta y)$ solves $x_i = 1 + \sum_{k \neq i} Q_{ik} x_k$ for all i. By the definition (and uniqueness) of the harmonic distance, that implies that

$$d_{j \to i} = \frac{D_i}{\ell - \theta y}. \tag{71}$$

As $D_i < F$ by hypothesis, $d_{j \to i} < F/(\ell - \theta y) \equiv d^{LT}$, as desired. □

A.10 Proof of Proposition 5 (Bottleneck connectedness)

We prove the two statements of the lemma sequentially. They rely on Lemma A.5 above (which, recall, depends only on the network structure, not the maturity of debt despite being stated within the short-term debt benchmark).

Proof of statement (i). For $\beta > \beta^{LT} \equiv 4\sqrt{\frac{\ell - \theta y}{NF}}$, we have, from Lemma A.5, that

$$d_{j \to i} \leq \frac{16}{N\beta^2} < \frac{16}{N(\beta^{LT})^2} = \frac{F}{\ell - \theta y} \equiv d^{LT}, \tag{72}$$

per the definition of d^{LT} in Proposition 4. That result implies that when B_j is not shocked, then B_i is not liquidated.

Proof of statement (ii). For $\beta < \beta_{LT} \equiv \min\left\{\frac{\ell - \theta y}{2NF}, 1\right\}$, we have, from Lemma A.5, that, for some i,

$$d_{j\to i} \geq \frac{1}{2N\beta} > \frac{1}{2N\beta_{LT}} = \max\left\{\frac{F}{\ell-\theta y}, \frac{1}{2N}\right\} \equiv \max\left\{d^{LT}, \frac{1}{2N}\right\} \geq d^{LT}, \tag{73}$$

per the definition of d^{LT} in Proposition 4. That result implies that when B_j is not shocked, then B_i does not default and, hence, not all banks are liquidated. □

A.11 Proof of Lemma 6 (Constrained efficiency)

The argument is in the text.

A.12 Proof of Lemma 7 (High debt mutualizes assets)

To prove the result, we show that if a not-shocked bank, say B_i, makes net payment less than θy no matter how high α is, it is impossible that another bank, say B_j, is liquidated. From equation (5), the not-shocked bank's net payment is

$$R_{i\rightrightarrows} - R_{i\leftleftarrows} = \begin{cases} \theta y & \text{if defaults,} \\ \alpha F_{i\rightrightarrows} - R_{i\leftleftarrows} & \text{otherwise.} \end{cases} \tag{74}$$

Now suppose (in anticipation of a contradiction) that B_i's payment is strictly less than θy and another bank, say B_j is liquidated. From equation (2), that implies that

$$\begin{aligned} \ell &> \theta y + R_{j\leftleftarrows} && (75)\\ &= \theta y + \sum_{k\neq i,j} R_{k\to j} + R_{i\to j} && (76)\\ &= \theta y + \sum_{k\neq i,j} R_{k\to j} + \alpha F_{i\to j}, && (77) \end{aligned}$$

having used the fact that B_i repays B_j in full (otherwise its net payment would equal θy). The inequality cannot hold for large α (as $F_{i\to j} > 0$ by the assumption that the network is connected). Thus either no bank is liquidated or B_i's net payment cannot be less than θy. $\square$

A.13 Proof of Corollary 1 (First best with high debt for small shocks)

Suppose (in anticipation of a contradiction) that at least one bank is liquidated. From Lemma 7 and equation (4) we know that for each B_i the net payment is

$$R_{i\rightrightarrows} - R_{i\leftleftarrows} \begin{cases} = \theta y - \ell\sigma_i & \text{if } \mathrm{B}_i \text{ is not liquidated,} \\ > \theta y - \ell\sigma_i & \text{if } \mathrm{B}_i \text{ is liquidated.} \end{cases} \tag{78}$$

Combining market clearing (equation (6)) with the expression above, we have that

$$0 = \sum \left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) > \sum \left(\theta y - \ell\sigma_i\right) = N\theta y - S\ell, \tag{79}$$

contradicting the hypothesis that $N\theta y > S\ell$. Therefore no bank can be liquidated, as desired.

A.14 Proof of Lemma 8 (Controlling relative payments to liquidated banks)

Here we use the pro rata condition that $R_{i\to j} = \hat{F}_{i\to j} R_{i\rightrightarrows}$ for any j (including $j = i^*$) to write

$$R_{i\to j} = \frac{\hat{F}_{i\to j}}{\hat{F}_{i\to i^*}} R_{i\to i^*}. \tag{80}$$

Thus the total payment to any liquidated bank B_j is

$$R_{j\leftleftarrows} = \sum_{i\neq j,i^*} R_{i\to j} + R_{i^*\to j} \tag{81}$$

$$= \sum_{i\neq j,i^*} \frac{\hat{F}_{i\to j}}{\hat{F}_{i\to i^*}} R_{i\to i^*}, \tag{82}$$

having used that $R_{i^*\to j} = 0$ as any liquidated bank makes zero repayment (equation (4)). Building on the above by substituting from the definition of s-dominance and adding the non-negative term $s^{j-i^*} R_{j\to i^*}$ gives

$$R_{j\leftleftarrows} \leq \sum_{i\neq j,i^*} s^{j-i^*} R_{i\to i^*} + s^{j-i^*} R_{j\to i^*} = s^{j-i^*} R_{i^*\leftleftarrows}. \tag{83}$$

□

A.15 Proof of Lemma 9 (Controlling total payments to liquidated banks)

Since liquidated banks make zero repayments, $R_{i\rightrightarrows} = 0$ for $i \in \mathscr{L}$, equation (4) implies that each receives payment $R_{i\leftleftarrows} < \ell - \theta y$. Applying this to B_{i^*}, the largest liquidated bank, and using Lemma 8, we have the following:

$$\sum_{i\in\mathscr{L}} R_{i\leftleftarrows} \leq \sum_{i\in\mathscr{L}} s^{i-i^*} R_{i^*\leftleftarrows} \tag{84}$$

$$\leq R_{i^*\leftleftarrows} \sum_{i=0}^{\infty} s^i \tag{85}$$

$$= R_{i^*\leftleftarrows} \frac{1}{1-s} \tag{86}$$

$$< \frac{\ell - \theta y}{1-s}. \tag{87}$$

□

A.16 Proof of Proposition 6 (Efficiency of exponential networks)

As the exponential network is connected, we know from Lemma 6 and Corollary 1 that if $L^* = 0$ then no bank is liquidated as long as α is high. Hence we focus on the case in which at least one bank is liquidated in the constrained-efficient outcome, $L^* \geq 1$.

Recall that in this case it suffices to show the following:

(i) Each bank that is not liquidated makes the maximum net payment it can without being liquidated, $R_{i\rightrightarrows} - R_{i\leftleftarrows} = \theta y - \ell\sigma_i$, for B_i not liquidated.

(ii) The banks that are liquidated receive a total net payment that would be insufficient to save any one of them, $-\sum_{i\in\mathscr{L}}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) < \ell - \theta y$.

The first property follows from Lemma 7 and equation (4).

The second property follows from two steps. The first is to use Lemma 9 and the definition of s^* to bound the liquidated banks' net payment in terms of L^*:

$$-\sum_{i\in\mathscr{L}}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) < \frac{\ell - \theta y}{1-s} \tag{88}$$

$$\leq \frac{\ell - \theta y}{1-s^*} \tag{89}$$

$$= N\theta y - S\ell + (1+L^*)(\ell - \theta y). \tag{90}$$

The second step is to use market clearing, $\sum_{i=1}^{N}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) = 0$ by equation (6), to write the LHS above in terms of the number of liquidated banks L, using that (i) not-shocked banks make net payment θy (by Lemma 7) and (ii) shocked, not-liquidated banks make net payment $\theta y - \ell$ (by equation 4):

$$-\sum_{i \in \mathscr{L}} \left(R_{i \rightrightarrows} - R_{i \leftleftarrows}\right) = \sum_{i \in \mathscr{L}^c} \left(R_{i \rightrightarrows} - R_{i \leftleftarrows}\right) \tag{91}$$

$$= \sum_{i \in \mathscr{L}^c : \sigma_i = 0} \left(R_{i \rightrightarrows} - R_{i \leftleftarrows}\right) + \sum_{i \in \mathscr{L}^c : \sigma_i = 1} \left(R_{i \rightrightarrows} - R_{i \leftleftarrows}\right) \tag{92}$$

$$= (N - S)\theta y + (S - L)(\theta y - \ell). \tag{93}$$

Combining this with the bound in equation (90) and canceling terms says $L < 1 + L^*$. As L and L^* are integers, and $L^* \leq L$ by Lemma 6, it must be that $L = L^*$.

(The assumption that $\frac{S\ell - N\theta y}{\ell - \theta y}$ not be an integer was required for for $s^* > 0$ and thus for the exponential network with base $s < s^*$ to be well defined.) □

A.17 Proof of Proposition 7 (Approximate efficiency of exponential networks)

With the weaker notation of efficiency, we need to show only that the banks that are liquidated receive a total net payment insufficient to save any *two* of them, $-\sum_{i \in \mathscr{L}} \left(R_{i \rightrightarrows} - R_{i \leftleftarrows}\right) < 2\big(\ell - \theta y\big)$. Given the proof of Proposition 6, that is all we need to show. As in that proof, it follows Lemma 9 along with the definitions of $s^* (= 1/2)$ and L^*:

$$-\sum_{i \in \mathscr{L}} \left(R_{i \rightrightarrows} - R_{i \leftleftarrows}\right) < \frac{\ell - \theta y}{1 - s^*} \tag{94}$$

$$= 2(\ell - \theta y) \tag{95}$$

as desired. □

A.18 Proof of Lemma 10 (Inefficiency of other networks)

Suppose (in anticipation of a contradiction) that a fully connected network $\mathbf{F}$ achieves constrained efficiency, i.e. that the number of liquidated banks is $L^* = \frac{S\ell - N\theta y}{\ell - \theta y}$, having used the assumptions $\frac{S\ell - N\theta y}{\ell - \theta y}$ is an integer and that $S\ell > N\theta y$ in conjunction with the definition of L^* (equation (10)).

As each banks that is not shocked pays at most θy and each that is shocked but not liquidated pays exactly $\theta y - \ell\sigma_i$ (equation (4)), we can use market clearing to bound the total payment to the liquidated banks as follows:

$$
\begin{aligned}
-\sum_{i\in\mathscr{L}}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) &= \sum_{i\notin\mathscr{L}}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) && (96)\\
&\leq \sum_{i\notin\mathscr{L}}(\theta y - \sigma_i\ell) && (97)\\
&= (N-S)\theta y + (S-L^*)(\theta y - \ell) && (98)\\
&= 0. && (99)
\end{aligned}
$$

I.e. liquidated banks receive no (positive) payment.

But shocked, not liquidated banks must receive positive payment (otherwise they would be liquidated by equation (4)). Given the hypothesis that the network is fully connected, that contradicts the assumption that payments are pro rata (equation (1)): Any bank that has debt to a not-liquidated bank must have debt to a liquidated bank too and it cannot make a positive payment to one but not the other. □

B Notations

To the extent possible, we use bold face letters for matrices and vectors and italics for scalars; we use single-arrow subscripts for debts from one bank to another and double-arrow subscripts for total debts from one to many banks. E.g., $\mathbf{F} = [F_{i\to j}]_{ij}$ is the matrix of interbank debts between individual banks; $\mathbf{F}_{\rightrightarrows} = [F_{i\rightrightarrows}]_i$ is the vector of banks' total interbank debts, i.e. the vector of row sums of $\mathbf{F}$. We use B_i for individual banks and script letters for sets; $\mathrm{B}_i \in \mathscr{B}$ and $i \in \mathscr{B}$ are synonymous. We summarize our notations in Table 2, separating those used in the main text from those used only in the proofs.

Table 2: Notations.

Notation	Meaning	Parametric restriction
y	Long-term real asset value	$y > 0$
ℓ	Size of liquidity shock	$\theta y < \ell < y$
θ	Pledgeable fraction of y	$0 < \theta < 1$
σ_i	Indicator if B_i's shock	$\sigma_i \in \{0, 1\}$
$\boldsymbol{\sigma} \equiv \{\sigma_i\}_i$	Vector of shocks/Aggregate state	$\boldsymbol{\sigma} \in \{0, 1\}^N$
B_i	ith bank	
$\mathscr{B}$	A set of banks	
$\mathscr{B}^c$	Complement of $\mathscr{B}$	
$\mathscr{L}$	Set of banks that are liquidated	$\mathscr{L} \subset \mathscr{D}$
N	Number of banks	
$S = \sum \sigma_i$	Number of shocked banks	
$L = \lvert\mathscr{L}\rvert$	Number of liquidated banks	
L^*	Minimum L (Lemma 6)	
$F_{i\to j}$	B_i's debt to B_j	$F_{i\to j} \geq 0$
$\mathbf{F} \equiv [F_{i\to j}]_{ij}$	Matrix of interbank debts	
$F_{i\rightrightarrows} \equiv \sum_{j\neq i} F_{i\to j}$	B_i's total debts to other banks	
F	Each bank's total debt $F \equiv F_{i\rightrightarrows}$ in a regular network	
$F_{i\leftleftarrows} \equiv \sum_{j\neq i} F_{j\to i}$	B_i's total debts from other banks	
$\mathbf{F}_{\rightrightarrows} \equiv \{F_{i\rightrightarrows}\}_i$	Vector of each bank's total interbank debt	
$\hat{F}_{i\to j} \equiv F_{i\to j}/F_{i\rightrightarrows}$	B_i's debt to B_j as a fraction of its total debts	$0 \leq \hat{F}_{i\to j} \leq 1$
$\hat{\mathbf{F}} \equiv [\hat{F}_{i\to j}]_{ij}$	Matrix of interbank debts	$\sum_i \hat{F}_{i\to j} = 1$
$R_{i\to j}$	B_i's equilibrium repayment to B_j	$0 \leq R_{i\to j} \leq F_{i\to j}$
$R_{i\rightrightarrows} \equiv \sum_{j\neq i} R_{i\to j}$	B_i's total repayment to other banks	
$R_{i\leftleftarrows} \equiv \sum_{j\neq i} F_{j\to i}$	B_i's total repayment received from other banks	

Continued on next page

Notations (continued)

Notation	Meaning	Parametric restriction
$\mathbf{R}_{\rightrightarrows} \equiv \{R_{i\rightrightarrows}\}_i$	Vector of each bank's total equilibrium repayment	$\mathbf{0} \leq \mathbf{R}_{\rightrightarrows} \leq \mathbf{F}_{\rightrightarrows}$
$\mathbf{R}_{\leftleftarrows} \equiv \{R_{i\leftleftarrows}\}_i$	Vector of each bank's total payment received	$\mathbf{R}_{\leftleftarrows} = \hat{\mathbf{F}}^{\top}\mathbf{R}_{\rightrightarrows}$
α	Scale of debts used in, e.g., Proposition 2	$\alpha > 0$
β	Bottleneck parameter (Definition 6)	
$d_{i\to j}$	Harmonic distance from B_i to B_j (Definition 5)	$d_{i\to j} \geq 0$
δ	Connectedness parameter (Definition 4)	$0 < \delta < 1$
d^{ST}, d^{LT}	Default and salvation radii in Lemma 4 and Proposition 4	
$\beta^{ST}, \beta_{ST}, \beta^{LT}, \beta_{LT}$	Thresholds in Lemma 5 and Proposition 5	
s	Dominance parameter (Definition 9)	$0 < s < 1$
s^*	Threshold in Proposition 6	
t_i	Transfer to B_i in Definition 7	
i^*	Index of largest liquidated bank in Section 5	
π_i	Permutation of banks in Definition	
y^*	Efficient liquidation threshold in Section 6.1	
$[\cdot]^+ = \max\{\cdot, 0\}$	Maximum of variable and zero	
$\lceil\cdot\rceil, \lfloor\cdot\rfloor$	Ceiling and floor functions	

Notations Used Only in Proofs	
$\mathbf{Q}$	A matrix, usually shorthand for $\hat{\mathbf{F}}^{\top}$
$\mathbf{Q}_{\mathscr{B}_1,\mathscr{B}_2} \equiv [Q_{ij}]_{i\in\mathscr{B}_1, j\in\mathscr{B}_2}$	Block matrix with rows in $\mathscr{B}_1$ and columns in $\mathscr{B}_2$
$\tilde{\mathbf{Q}}$	Matrix in Lemma A.8
$\mathbf{0}, \mathbf{1}$	Vectors of zeros and ones ($(0, ..., 0)$ and $(1, ..., 1)$)
$\mathbf{d}_{i\to\mathscr{B}} \equiv \{d_{i\to j}\}_{j\in\mathscr{B}}$	Vector of B_i's harmonic distance to banks in $\mathscr{B}$
$D_i \equiv F_{i\rightrightarrows} - R_{i\rightrightarrows}$	B_i's shortfall
$\mathbf{D} \equiv \mathbf{F}_{\rightrightarrows} - \mathbf{R}_{\rightrightarrows}$	Vector of each bank's shortfall
$\mathscr{D}$	Set of banks that default
$\Phi^{\alpha}, \Psi^{\alpha}$	Mappings used in Lemma A.1 and Lemma A.6
$\mathscr{H}^{\alpha}, \mathscr{I}^{\alpha}$	Restricted domains of Φ^{α} and Ψ^{α}

Continued on next page

Notations (continued)

Notation	Meaning	Parametric restriction
$\prod_{i=1}^{N} X_i = X_1 \times \cdots \times X_N$	Cartesian product of sets $X_1, ..., X_N$	

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